Answers to problems

Problem 2.1 The integrand in equation (2.53) may be expressed as

$$\rho\,\mathbf{v}\cdot \mathrm{D}_t\mathbf{v} + \rho\,\mathrm{D}_t\mathrm{u} + \boldsymbol{\nabla}\cdot\mathbf{H} - \boldsymbol{\nabla}\cdot(\mathbf{v}\cdot\boldsymbol{\sigma}) - \mathrm{h} - \mathbf{f}\cdot\mathbf{v} = 0\,.$$

Writing the term $\boldsymbol{\nabla}\cdot(\mathbf{v}\cdot\boldsymbol{\sigma})$ in index notation, we have

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\mathbf{v}\cdot\boldsymbol{\sigma}) &= \nabla_i(v^j\,\sigma_j{}^i)\\ &= v^j\,\nabla_i\sigma_j{}^i + \sigma_j{}^i\,\nabla_i v^j\,.\\ &= \mathbf{v}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}) + \boldsymbol{\sigma}:\mathbf{G}\,.\end{aligned}$$

Thus the full expression may be rewritten in the form

$$\mathbf{v}\cdot(\rho\,\mathrm{D}_t\mathbf{v} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \mathbf{f}) + \rho\,\mathrm{D}_t\mathrm{u} + \boldsymbol{\nabla}\cdot\mathbf{H} - \boldsymbol{\sigma}:\mathbf{G} - \mathrm{h} = 0\,,$$

in agreement with equation (2.54).

Problem 2.2 Conservation of energy (2.56) is readily translated in Eulerian coordinates and index notation as

$$\rho\,(\partial_t u + v^i\,\nabla_i u) + \nabla_i H^i = \sigma^i{}_j\,G_i{}^j + h\,.$$

To obtain the Lagrangian version, we multiply by the Jacobian J and use the relationship $\mathring{\varrho}\,U = J\rho u$ and define $\mathring{h} \equiv J\,h$, and $\mathring{H}^I \equiv J\,H^i\,(F^{-1})^I{}_i = J\,H^I$. We find

$$J\,\rho\,(\partial_t u + v^i\,\nabla_i u) + J\,\nabla_i H^i = J\,\sigma^i{}_j\,G_i{}^j + \mathring{h}\,.$$

Next, we use the fact that

$$\mathring{\varrho}\,\partial_T U = J\,\rho\,(\partial_t u + v^i\,\nabla_i u)$$

such that we have

$$\mathring{\varrho}\,\partial_T U + J\,\nabla_i H^i = J\,\sigma^i{}_j\,G_i{}^j + \mathring{h}\,.$$

Next, we use expression (1.92) and relationship (2.12) between the Cauchy stress and the first Piola-Kirchhoff stress to write

$$\begin{aligned}J\,\sigma^i{}_j\,G_i{}^j &= J\,\sigma_i{}^j\,G^i{}_j\\ &= J\,\sigma_i{}^j\,(F^{-1})^I{}_j\,\partial_T F^i{}_I\\ &= P_i{}^I\,\partial_T F^i{}_I\,,\end{aligned}$$

and so

$$\mathring{\varrho}\,\partial_T U + J\,\nabla_i H^i = P_i{}^I\,\partial_T F^i{}_I + \mathring{h}\,.$$

Finally, we have

$$\begin{aligned}J\,\nabla_i H^i &= J\,\nabla_I H^I\\ &= J\,\partial_I H^I + J\,\Gamma^J_{JI}\,H^I\\ &= \partial_I(J\,H^I) - H^I\,\partial_I J + J\,\Gamma^J_{JI}\,H^I\\ &= \partial_I\mathring{H}^I + (\Gamma^J_{JI} - J^{-1}\,\partial_I J)\,\mathring{H}^I\\ &= \partial_I\mathring{H}^I + \Gamma^J_{JI}\,\mathring{H}^I\end{aligned}$$

where in the last equality we used (1.168). Thus we finally obtain the desired result:

$$\mathring{\varrho}\,\partial_T U + \mathring{\nabla}_I \mathring{H}^I = P_i{}^I\,\partial_T F^i{}_I + \mathring{h}\,,$$

where $\mathring{\nabla}_I H^I = \partial_I \mathring{H}^I + \mathring{\Gamma}^I_{IJ}\,\mathring{H}^J$.

Problem 2.3 From equation (2.97) we find that

$$\begin{aligned}\frac{\partial g_{KL}}{\partial \partial_I \varphi^i} &= g_{k\ell}\,\frac{\partial \partial_K \varphi^k}{\partial \partial_I \varphi^i}\,\partial_L \varphi^\ell + g_{k\ell}\,\partial_K \varphi^k\,\frac{\partial \partial_L \varphi^\ell}{\partial \partial_I \varphi^i}\\ &= g_{k\ell}\,\delta^I{}_K\,\delta^k{}_i\,\partial_L \varphi^\ell + g_{k\ell}\,\partial_K \varphi^k\;\delta^I{}_L\,\delta^\ell{}_i\\ &= g_{ij}\,\delta^I{}_K\;\partial_L \varphi^j + g_{ij}\,\partial_K \varphi^j\;\delta^I{}_L\\ &= (\delta^I{}_K\,\delta^J{}_L + \delta^I{}_L\,\delta^J{}_K)\,g_{ij}\,\partial_J \varphi^j\,,\end{aligned}$$

in agreement with (2.98).

Problem 2.4 Upon dividing equation (2.102) by the Jacobian of the motion J and using equations (1.238), (2.12), and (1.98), we find

$$P\,\partial_T U = 2\,P\left(\frac{\partial U}{\partial g_{IJ}}\right)_S \partial_T g_{IJ} = \tfrac{1}{2}\,\sigma^{IJ}\,\partial_T g_{IJ}\,,$$

where σ^{IJ} are defined by (2.106). The tensor version of this expression is equation (2.105).

Problem 2.5. Expressed in Lagrangian components, equation (2.109) is

$$\partial_T \sigma^{IJ} + (\nabla_K v^K)\,\sigma^{IJ} = a^{IJKL}\,D_{KL}\,.$$

According to (F.154), in Lagrangian coordinates,

$$\begin{aligned}(\mathcal{L}_{\mathbf{v}}\boldsymbol{\sigma}^\sharp)^{IJ} = \partial_T \sigma^{IJ} &= (\mathrm{D_t}\boldsymbol{\sigma}^\sharp)^{IJ} - \sigma^{KJ}\,\nabla_K v^I - \sigma^{IK}\,\nabla_K v^J\\ &= (\mathrm{D_t}\boldsymbol{\sigma}^\sharp)^{IJ} - \sigma^{KJ}\,G^I{}_K - \boldsymbol{\sigma}^{IK}\;G^J{}_K\,,\end{aligned}$$

and thus

$$(\mathrm{D_t}\boldsymbol{\sigma}^\sharp)^{IJ} - G^I{}_K\,\sigma^{KJ} - \sigma^{IK}\;G^J{}_K + (\nabla_K v^K)\,\sigma^{IJ} = a^{IJKL}\,D_{KL}\,.$$

The tensor form of this is equation (2.111). Using the decomposition (1.84) and the definition of the Zaremba-Jaumann rate (1.118) then leads to equation (2.112).

Problem 2.6. Setting $\mathbf{D} = \mathbf{0}$ in equation (2.112) immediately leads to the desired expression (2.113).

Problem 2.7. Upon substituting relationship (2.6) into equation (2.114), we find

$$\begin{aligned}\partial_T(J^{-1}\,\tau^{IJ}) + (\nabla_K v^K)\,(J^{-1}\,\tau^{IJ}) &= J^{-1}\,\partial_T \tau^{IJ} - J^{-2}\,\tau^{IJ}\,\partial_T J + (\nabla_K v^K)\,(J^{-1}\,\tau^{IJ})\\ &= J^{-1}\,\partial_T \tau^{IJ}\\ &= a^{IJKL}\,D_{KL}\,,\end{aligned}$$

where we used the evolution equation (1.164). This result is in agreement with equation (2.115).

Problem 2.8. Upon setting $\mathbf{D} = \mathbf{0}$ in equation (2.118), we find that

$$\mathcal{L}_{\mathbf{v}}\mathbf{a} = \mathbf{0}\,.$$

According to (F.154), in Lagrangian coordinates,

$$
\begin{aligned}
(\mathcal{L}_{\mathbf{v}}\mathbf{a})^{IJKL} &= \partial_T a^{IJKL} \\
&= (\mathrm{D_t}\mathbf{a})^{IJKL} - a^{MJKL}\,\nabla_M v^I - a^{IMKL}\,\nabla_M v^J - a^{IJML}\,\nabla_M v^K - a^{IJKM}\,\nabla_M v^L \\
&= (\mathrm{D_t}\mathbf{a})^{IJKL} - a^{MJKL}\,G^I{}_M - a^{IMKL}\,G^J{}_M - a^{IJML}\,G^K{}_M - a^{IJKM}\,G^L{}_M\,.
\end{aligned}
$$

Upon setting $\mathbf{D} = \mathbf{0}$ in this expression, we find that

$$
\begin{aligned}
(\mathcal{L}_{\mathbf{v}}\mathbf{a})^{IJKL} &= (\mathrm{D_t}\mathbf{a})^{IJKL} - a^{MJKL}\,W^I{}_M - a^{IMKL}\,W^J{}_M - a^{IJML}\,W^K{}_M - a^{IJKM}\,W^L{}_M \\
&= ({}^{\mathrm{J}}\mathbf{a})^{IJKL} \\
&= 0\,,
\end{aligned}
$$

in agreement with equation (2.121).

Problem 2.9. As stated after equation (2.125), the assumption is that the elements of the elastic tensor Ξ^{IJKL} are independent of the convected time T. Thus, trivially, $\partial_T \Xi^{IJKL} = 0$, which has the tensor expression (2.133).

Problem 2.10. We have, using (2.155), (2.156), and (1.327),

$$
\begin{aligned}
\boldsymbol{\sigma}^i &= \mathbf{c}^{ij} : \mathbf{E}_j \\
&= \tfrac{1}{8}\, c^{ij}{}_{k\ell mn}\,\mathbf{e}^k \wedge \mathbf{e}^\ell \otimes \mathbf{e}^m \wedge \mathbf{e}^n : \tfrac{1}{2}\,E_{jp}\,\epsilon^{pqr}\,\mathbf{e}_q \wedge \mathbf{e}_r \\
&= \tfrac{1}{8}\, c^{ij}{}_{k\ell mn}\,E_{jp}\,\epsilon^{pqr}\,(\delta^m{}_q\,\delta^n{}_r - \delta^n{}_q\,\delta^m{}_r)\,\mathbf{e}^k \wedge \mathbf{e}^\ell \\
&= \tfrac{1}{4}\, c^{ij}{}_{k\ell mn}\,E_{jp}\,\epsilon^{pmn}\,\mathbf{e}^k \wedge \mathbf{e}^\ell \\
&= \tfrac{1}{4}\, c^{ipjq}\,\epsilon_{pk\ell}\,\epsilon_{qmn}\,E_{jr}\,\epsilon^{rmn}\,\mathbf{e}^k \wedge \mathbf{e}^\ell \\
&= \tfrac{1}{2}\, c^{ipjq}\,\epsilon_{pk\ell}\,\delta^r{}_q\,E_{jr}\,\mathbf{e}^k \wedge \mathbf{e}^\ell \\
&= \tfrac{1}{2}\, \sigma^{ip}\,\epsilon_{pk\ell}\,\mathbf{e}^k \wedge \mathbf{e}^\ell\,.
\end{aligned}
$$

Problem 2.11. Upon dividing the Euler-Lagrange equation (2.177) by the Jacobian of the motion J, using the Lagrangian version of conservation of mass (1.238), and property (1.168), we find

$$
\begin{aligned}
&J^{-1}(\rho^0\,\partial_T^2\varphi^i - \partial_I P^{iI} - P^{iI}\,\Gamma^{0\,K}_{IK}) \\
&\quad= P\,\partial_T^2\varphi^i - \partial_I(J^{-1}\,P^{iI}) - J^{-2}\,P^{iI}\,\partial_I J - J^{-1}P^{iI}\,\Gamma^{0\,K}_{IK} \\
&\quad= P\,\partial_T^2\varphi^i - \partial_I(J^{-1}\,P^{iI}) - J^{-1}\,P^{iI}\,(\Gamma^K_{IK} - \Gamma^{0\,K}_{IK}) - J^{-1}P^{iI}\,\Gamma^{0\,K}_{IK} \\
&\quad= P\,\partial_T^2\varphi^i - \partial_I(J^{-1}\,P^{iI}) - J^{-1}P^{iI}\,\Gamma^K_{IK}\,.
\end{aligned}
$$

Next, upon substituting relationship (2.180) between the first Piola-Kirchhoff stress and the Cauchy stress we obtain

$$
\begin{aligned}
&P\,\partial_T^2\varphi^i - \partial_I(J^{-1}\,P^{iI}) - J^{-1}P^{iI}\,\Gamma^K_{IK} \\
&\quad= \rho\,(\partial_t v^i + v^j\,\partial_j v^i) - F^i{}_I\,\partial_J\sigma^{IJ} - \sigma^{IJ}\,\partial_J F^i{}_I - F^i{}_I\,\sigma^{IJ}\,\Gamma^K_{JK} \\
&\quad= \rho\,(\partial_t v^i + v^j\,\partial_j v^i) - F^i{}_I\,\partial_J\sigma^{IJ} - \sigma^{KJ}\,F^i{}_I\,(F^{-1})^I{}_k\partial_J F^k{}_K - F^i{}_I\,\sigma^{IJ}\,\Gamma^K_{JK} \\
&\quad= \rho\,(\partial_t v^i + v^j\,\partial_j v^i) - F^i{}_I\,(\partial_J\sigma^{IJ} + \sigma^{KJ}\,\Gamma^I_{JK} + \sigma^{IJ}\,\Gamma^K_{JK}) \\
&\quad= \rho\,(\partial_t v^i + v^j\,\partial_j v^i) - F^i{}_I\,\nabla_J\sigma^{IJ} \\
&\quad= \rho\,(\partial_t v^i + v^j\,\partial_j v^i) - \partial_j\sigma^{ij} \\
&\quad= 0\,,
\end{aligned}
$$

where we have used equations (1.48) and the definition of the covariant derivative.
Problem 2.12. Using Nanson's relation (2.183) in (2.178) yields

$$\begin{aligned}\left[P^{iI}\,\hat{n}_I\right]_-^+\,d\mathring{\mathrm{S}} &= \left[J^{-1}\,F^j{}_I\,P^{iI}\,\hat{n}_j\right]_-^+\,d\mathrm{S}\\ &= \left[\sigma^{ij}\,\hat{n}_j\right]_-^+\,d\mathrm{S}\,.\end{aligned}$$

The tensor form is (2.183).
Problem 2.13.

We start with (2.199), the Eulerian form of conservation of linear momentum, namely

$$\partial_t(\rho\,v^i) = \nabla_j(\sigma^{ij} - \rho\,v^i\,v^j) + \rho\,g^i\,.$$

The Euler derivative of the mass flux in Eulerian spatial components is given by

$$[\mathrm{d_t}(\rho\,\mathbf{v})]^i = \partial_t(\rho\,v^i)\,.$$

We use the Euler derivative in Lagrangian spatial components (1.110) for the mass flux $\rho\,\mathbf{v}$:

$$[\mathrm{d_t}(\rho\,\mathbf{v})]^I = \partial_T(\varrho\,v^I) - v^J\,\nabla_J(\varrho\,v^I) + \varrho\,v^J\,\nabla_J v^I\,.$$

Using this transformation, we may rewrite conservation of linear momentum in the Lagrangian form

$$\partial_T(\varrho\,v^I) - v^J\,\nabla_J(\varrho\,v^I) + \varrho\,v^J\,\nabla_J v^I = \nabla_J(\sigma^{IJ} - \varrho\,v^I\,v^J) + \varrho\,g^I\,.$$

Rearranging these terms leads to the desired expression:

$$\partial_T(\varrho\,v^I) + \varrho\,(\nabla_J v^I)\,v^J + \varrho\,v^I\,\nabla_J v^J = \nabla_J\sigma^{IJ} + \varrho\,g^I\,.$$

Problem 2.14. According to (1.214), we have

$$\begin{aligned}\mathfrak{L}_{\mathbf{v}}\mathbf{p}^I &= \mathcal{L}_{\mathbf{v}}\mathbf{p}^I + \mathbf{p}^J\,\nabla_J V^I\\ &= \mathcal{L}_{\mathbf{v}}(\varrho\,v^I\,\boldsymbol{\epsilon}) + \varrho\,v^J\,\nabla_J V^I\,\boldsymbol{\epsilon}\\ &= \varrho\,v^I\,\mathcal{L}_{\mathbf{v}}\boldsymbol{\epsilon} + [\partial_T(\varrho\,v^I) + \varrho\,v^J\,\nabla_J V^I]\,\boldsymbol{\epsilon}\\ &= [\partial_T(\varrho\,v^I) + \varrho\,v^J\,\nabla_J V^I + \varrho\,v^I\,\nabla_J v^J]\,\boldsymbol{\epsilon}\,.\end{aligned}$$

This is the left-hand side of expression (2.202). We also have

$$\begin{aligned}\mathrm{D}\boldsymbol{\sigma}^I &= \mathrm{d}\boldsymbol{\sigma}^I + \boldsymbol{\Gamma}^I_J\wedge\boldsymbol{\sigma}^J\\ &= \mathrm{d}\left(\tfrac{1}{2}\,\sigma^{IJ}\,\epsilon_{JKL}\,\mathbf{e}^K\wedge\mathbf{e}^L\right) + \tfrac{1}{2}\,\Gamma^I_{NJ}\,\sigma^{JM}\,\epsilon_{MKL}\,\mathbf{e}^N\wedge\mathbf{e}^K\wedge\mathbf{e}^L\\ &= \tfrac{1}{2}\,\partial_M\left(\sigma^{IJ}\,\overline{G}\right)\,\underline{G}\,\epsilon_{JKL}\,\mathbf{e}^M\wedge\mathbf{e}^K\wedge\mathbf{e}^L + \Gamma^I_{NJ}\,\sigma^{JM}\,\delta_M{}^N\,\boldsymbol{\epsilon}\\ &= \left(\partial_J\sigma^{IJ} + \sigma^{IJ}\,\underline{G}\,\partial_J\overline{G} + \Gamma^I_{MJ}\,\sigma^{JM}\right)\,\boldsymbol{\epsilon}\\ &= \left(\partial_J\sigma^{IJ} + \sigma^{IJ}\,\Gamma^K_{JK} + \Gamma^I_{JM}\,\sigma^{MJ}\right)\,\boldsymbol{\epsilon}\\ &= \nabla_J\sigma^{IJ}\,\boldsymbol{\epsilon}\,,\end{aligned}$$

where we used relationships (1.136), (G.57) and (D.117). Finally, using $\mathbf{f}^I = \varrho\,g^I\,\boldsymbol{\epsilon}$, we obtain the desired result.

Problem 2.15. Expressed as a matrix, the determinant of equation (2.213) is

$$\begin{aligned}
\left|\frac{\partial X'^{\mu}}{\partial X^{\nu}}\right| &= \begin{vmatrix} 1+\epsilon^{\nu}\,\partial_0\lambda^0{}_{\nu} & \epsilon^{\nu}\,\partial_1\lambda^0{}_{\nu} & \epsilon^{\nu}\,\partial_2\lambda^0{}_{\nu} & \epsilon^{\nu}\,\partial_3\lambda^0{}_{\nu} \\ \epsilon^{\nu}\,\partial_0\lambda^1{}_{\nu} & 1+\epsilon^{\nu}\,\partial_1\lambda^1{}_{\nu} & \epsilon^{\nu}\,\partial_2\lambda^1{}_{\nu} & \epsilon^{\nu}\,\partial_3\lambda^1{}_{\nu} \\ \epsilon^{\nu}\,\partial_0\lambda^2{}_{\nu} & \epsilon^{\nu}\,\partial_1\lambda^2{}_{\nu} & 1+\epsilon^{\nu}\,\partial_2\lambda^2{}_{\nu} & \epsilon^{\nu}\,\partial_3\lambda^2{}_{\nu} \\ \epsilon^{\nu}\,\partial_0\lambda^3{}_{\nu} & \epsilon^{\nu}\,\partial_1\lambda^3{}_{\nu} & \epsilon^{\nu}\,\partial_2\lambda^3{}_{\nu} & 1+\epsilon^{\nu}\,\partial_3\lambda^3{}_{\nu} \end{vmatrix} \\
&\approx (1+\epsilon^{\nu}\,\partial_0\lambda^0{}_{\nu})(1+\epsilon^{\nu}\,\partial_1\lambda^1{}_{\nu})(1+\epsilon^{\nu}\,\partial_2\lambda^2{}_{\nu})(1+\epsilon^{\nu}\,\partial_3\lambda^3{}_{\nu}) \\
&\approx 1+\epsilon^{\nu}\,\partial_0\lambda^0{}_{\nu}+\epsilon^{\nu}\,\partial_1\lambda^1{}_{\nu}+\epsilon^{\nu}\,\partial_2\lambda^2{}_{\nu}+\epsilon^{\nu}\,\partial_3\lambda^3{}_{\nu} \\
&= 1+\epsilon^{\nu}\,\partial_{\mu}\lambda^{\mu}{}_{\nu}\,,
\end{aligned}$$

where we retained only first-order terms in ϵ^{ν}.

Problem 2.16. For a translation in space (2.210), we use $\lambda^0{}_{\nu}=0$, $\lambda^J{}_0=0$, and $\lambda^J{}_I=\delta^J{}_I$ in the generic change (2.212). Thus, setting $\nu=I$ in equation (2.223) we find

$$\partial_0 J_I{}^0+\partial_J J_I{}^J=0\,,$$

where the current density has components

$$J_I{}^0=\frac{\partial L}{\partial\partial_T\varphi^i}\,\partial_I\varphi^i\,,$$

$$J_I{}^J=\frac{\partial L}{\partial\partial_J\varphi^i}\,\partial_I\varphi^i-\delta^J{}_I\,L\,,$$

as stated.

Problem 2.17. We have

$$J_I{}^0=\frac{\partial L}{\partial\partial_T\varphi^i}\,\partial_I\varphi^i=\rho^0\,\partial_T\varphi^j\,\partial_I\varphi^i\,\delta_{ij}\,,$$

and so

$$\partial_T J_I{}^0=\rho^0\,\partial_T^2\varphi^j\,\partial_I\varphi^i\,\delta_{ij}+\rho^0\,\partial_T\varphi^j\,\partial_T\partial_I\varphi^i\,\delta_{ij}\,.$$

We also have

$$J_I{}^J=\frac{\partial L}{\partial\partial_J\varphi^i}\,\partial_I\varphi^i-\delta^J{}_I\,L=-\,P_i{}^J\,\partial_I\varphi^i-\delta^J{}_I\,\rho^0\,\left(\tfrac{1}{2}\,\partial_T\varphi^i\,\partial_T\varphi^j\,\delta_{ij}-U\right)\,,$$

and so

$$\begin{aligned}
\partial_J J_I{}^J &= -\,\partial_J P_i{}^J\,\partial_I\varphi^i-P_i{}^J\,\partial_J\partial_I\varphi^i-(\partial_I\rho^0)\,\left(\tfrac{1}{2}\,\partial_T\varphi^i\,\partial_T\varphi^j\,\delta_{ij}-U\right) \\
&\quad -\rho^0\,\partial_T\varphi^i\,\partial_I\,\partial_T\varphi^j\,\delta_{ij}+P_i{}^J\,\partial_J\partial_I\varphi^i+\rho^0\,\partial_I U \\
&= -\,\partial_J P_i{}^J\,\partial_I\varphi^i-(\partial_I\rho^0)\,\left(\tfrac{1}{2}\,\partial_T\varphi^i\,\partial_T\varphi^j\,\delta_{ij}-U\right) \\
&\quad -\rho^0\,\partial_T\varphi^i\,\partial_I\,\partial_T\varphi^j\,\delta_{ij}+\rho^0\,\partial_I U\,.
\end{aligned}$$

Upon combining the expressions for $\partial_T J_I{}^0$ and $\partial_J J_I{}^J$, we find equation (2.229).

Problem 2.18. For a translation in time (2.209), we use $\lambda^0{}_0=1$ in the generic change (2.212); all other elements vanish. Thus, setting $\nu=0$ in equation (2.223) we find

$$\partial_0 J_0{}^0+\partial_J J_0{}^J=0\,,$$

where the current density has components

$$J_0{}^0 = \frac{\partial L}{\partial \partial_T \varphi^i}\, \partial_T \varphi^i - L\,,$$

$$J_0{}^J = \frac{\partial L}{\partial \partial_I \varphi^i}\, \partial_T \varphi^i\,.$$

Problem 2.19. Start by writing out (2.268), assuming Cartesian Eulerian coordinates for simplicity

$$\tfrac{1}{2}\, \mathcal{L}_{\mathbf{v}}(v^2\, \boldsymbol{\rho}) + \mathcal{L}_{\mathbf{v}} \mathbf{e} = v^i\, \mathbf{f}_i + \mathbf{h} + \mathrm{d}v^i \wedge \boldsymbol{\sigma}_i + v^i\, \mathrm{d}\boldsymbol{\sigma}_i - \mathrm{d}\mathbf{H}\,,$$

where $\boldsymbol{\rho}$ denotes the mass three-form (1.224), which is conserved, (1.228). Thus, we have,

$$(\mathcal{L}_{\mathbf{v}} v_i)\, v^i\, \boldsymbol{\rho} + \mathcal{L}_{\mathbf{v}} \mathbf{e} = v^i\, \mathbf{f}_i + \mathbf{h} + \mathrm{d}v^i \wedge \boldsymbol{\sigma}_i + v^i\, \mathrm{d}\boldsymbol{\sigma}_i - \mathrm{d}\mathbf{H}\,,$$

or, again using conservation of mass, $\mathcal{L}_{\mathbf{v}} \boldsymbol{\rho} = \mathbf{0}\,,$

$$v^i\, (\mathcal{L}_{\mathbf{v}} \mathbf{p}_i - \mathrm{d}\boldsymbol{\sigma}_i - \mathbf{f}^i) + \mathcal{L}_{\mathbf{v}} \mathbf{e} = \mathrm{d}v^i \wedge \boldsymbol{\sigma}_i - \mathrm{d}\mathbf{H}\,,$$

where $\mathbf{p}_i = v_i\, \boldsymbol{\rho} = \rho\, v^i\, \boldsymbol{\epsilon}$. Upon using the form version of conservation of linear momentum (2.196), we obtain the desired result for conservation of energy.

Problem 2.20. Given the Lagrangian density (2.360), we have

$$\frac{\partial L}{\partial \partial_T \varphi^i} = \rho^0\, (\partial_T \varphi^j + \epsilon^j{}_{mn}\, \Omega^m\, \varphi^n)\, \delta_{ij}\,,$$

$$\frac{\partial L}{\partial \partial_I \varphi^i} = -\, P_i{}^I\,,$$

$$\frac{\partial L}{\partial \varphi^i} = \rho^0\, \epsilon^\ell{}_{ki}\, \Omega^k\, (\partial_T \varphi^j + \epsilon^j{}_{mn}\, \Omega^m\, \varphi^n)\, \delta_{\ell j}\,,$$

and thus the Euler-Lagrange equations (2.361) are

$$\begin{aligned}
\frac{\partial}{\partial T}\frac{\partial L}{\partial \partial_T \varphi^i} + \frac{\partial}{\partial X^I}\frac{\partial L}{\partial \partial_I \varphi^i} - \frac{\partial L}{\partial \varphi^i} &= \rho^0\, \partial_T^2 \varphi^j\, \delta_{ij} + \rho^0\, \epsilon_{imn}\, \Omega^m\, \partial_T \varphi^n - \partial_I P_i{}^I \\
&\qquad - \rho^0\, \epsilon_{jki}\, \Omega^k\, \partial_T \varphi^j - \rho^0\, \epsilon_{jki}\, \Omega^k\, \epsilon^j{}_{mn}\, \Omega^m\, \varphi^n \\
&= \rho^0\, \partial_T^2 \varphi^j\, \delta_{ij} + \rho^0\, \epsilon_{ijk}\, \Omega^j\, \partial_T \varphi^k - \partial_I P_i{}^I \\
&\qquad + \rho^0\, \epsilon_{ijk}\, \Omega^j\, \partial_T \varphi^k + \rho^0\, \epsilon_{ijk}\, \Omega^j\, \epsilon^k{}_{mn}\, \Omega^m\, \varphi^n \\
&= \rho^0\, (\partial_T^2 \varphi^j\, \delta_{ij} + 2\, \epsilon_{ijk}\, \Omega^j\, \partial_T \varphi^k + \epsilon_{ikj}\, \Omega^k\, \epsilon^j{}_{mn}\, \Omega^m\, \varphi^n) \\
&\qquad - \partial_I P_i{}^I\,,
\end{aligned}$$

in agreement with (2.362).

Problem 2.21. Upon taking the gradient of (2.365), we find, working in Cartesian spatial coordinates and using $r = (x^2 + y^2 + z^2)^{1/2}$ and $\nabla_i r = r_i / r\,,$

$$\begin{aligned}
\nabla_i \psi &= -\left[\Omega^2\, r_i - (\boldsymbol{\Omega} \cdot \mathbf{r})\, \Omega_i\right] \\
&= -\, (\delta_{i\ell}\, \delta_{jm} - \delta_{im}\, \delta_{j\ell})\, r_\ell\, \Omega_j\, \Omega_m \\
&= -\, \epsilon_{ijk}\, \epsilon_{k\ell m}\, r_\ell\, \Omega_j\, \Omega_m \\
&= [\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})]_i\,.
\end{aligned}$$

Problem 2.22 In Cartesian coordinates, we have

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2} .$$

Therefore, for example

$$\frac{\partial}{\partial x}\,\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = -\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-3/2} (x-x') .$$

Using this result, equation (2.372) follows directly by taking the negative gradient of equation (2.371).

Problem 2.23. To prove relationship (2.389), we find from equation (2.386) that

$$\frac{\partial \partial_K \varphi^k}{\partial \partial_J s^I} = \delta^k{}_L \,\frac{\partial \partial_K s^L}{\partial \partial_J s^I} = \delta^k{}_L \,\delta^J{}_K \,\delta^L{}_I = \delta^k{}_I \,\delta^J{}_K .$$

To prove relationship (2.390), we find from equation (2.387) that

$$\begin{aligned}
\frac{\partial g_{KL}}{\partial \partial_J s^I} &= 2\,\frac{\partial \partial_K s^M}{\partial \partial_J s^I}\,(\delta^N{}_L + \partial_L s^N)\,\delta_{MN} \\
&= 2\,\delta^J{}_K\,\delta^M{}_I\,(\delta^N{}_L + \partial_L s^N)\,\delta_{MN} \\
&= 2\,\delta^J{}_K\,\delta_{IL} + 2\,\delta^J{}_K\,\delta_{NI}\,\partial_L s^N .
\end{aligned}$$

Thus

$$\begin{aligned}
\rho^0\,\frac{\partial U}{\partial \partial_J s^I} &= \rho^0\,\frac{\partial U}{\partial g_{KL}}\,\frac{\partial g_{KL}}{\partial \partial_J s^I} \\
&= \tau^{KL}\,(\delta^J{}_K\,\delta_{IL} + \delta^J{}_K\,\delta_{NI}\,\partial_L s^N) \\
&= \delta_{IK}\,\tau^{KJ} + \tau^{KJ}\,\partial_K s^L\,\delta_{IL} ,
\end{aligned}$$

in agreement with (2.390).

Problem 2.24.

The variations related to rotation were determined in Problem 2.20, and the variations due to gravitation are given in Section 2.8.3. Combining these two contributions for the motion in terms of the displacement yields the desired results.