Answers to problems

Problem 5.1.

Option 1, illustrated in Figure 5.2 (*left*), combines a Maxwell solid in parallel with a spring. For the Maxwell solid, which is a spring and dashpot in series, the stresses in the elements are the same, σ, and we add the strain rate of the spring, $\dot{\sigma}/\delta M$, to the strain rate of the dashpot, $\sigma/\eta = \sigma/(\tau_\sigma\,\delta M)$, to obtain the overall strain rate $\dot{\epsilon} = (\dot{\sigma} + \sigma/\tau_\sigma)/\delta M$. To make the standard linear solid, we add a spring in parallel. According to the rules, this implies that the strain in the spring will be the same as that in the Maxwell solid. Let us label the stress in the Maxwell solid as σ_{M} and the total stress in the spring as σ_{S}. then we have $\dot{\epsilon} = (\dot{\sigma}_{\mathrm{M}} + \sigma_{\mathrm{M}}/\tau_\sigma)/\delta M$, $\epsilon = \sigma_{\mathrm{S}}/M_{\mathrm{r}}$, and $\sigma = \sigma_{\mathrm{M}} + \sigma_{\mathrm{S}}$. Eliminating σ_{S} we have $\epsilon = (\sigma - \sigma_{\mathrm{M}})/M_{\mathrm{r}}$, such that $\sigma_{\mathrm{M}} = \sigma - M_{\mathrm{r}}\,\epsilon$. Eliminating σ_{M}, we have $\dot{\epsilon} = (\dot{\sigma} - M_{\mathrm{r}}\,\dot{\epsilon} + \sigma/\tau_\sigma - M_{\mathrm{r}}\,\epsilon/\tau_\sigma)/\delta M$. This may be rearranged as

$$(M_{\mathrm{r}} + \delta M)\,\dot{\epsilon} + M_{\mathrm{r}}\,\epsilon/\tau_\sigma = \dot{\sigma} + \sigma/\tau_\sigma\,.$$

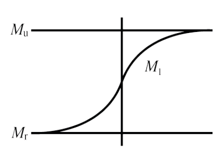
Using $M_{\mathrm{r}} + \delta M = M_{\mathrm{u}}$ and $M_{\mathrm{r}}/M_{\mathrm{u}} = \tau_\sigma/\tau_\epsilon$ then gives the desired result (5.10).

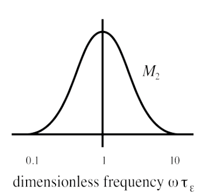
Option 2, illustrated in Figure 5.2 (*right*), combines a Kelvin-Voigt solid in series with a spring. For the Kelvin-Voigt solid, which is a spring and dashpot in parallel, the strains in the elements are the same, ϵ, and we add the stress of the spring, $\epsilon/\delta J$, to the stress of the dashpot, $\eta\,\dot{\epsilon} = \tau_\epsilon\,\dot{\epsilon}/\delta J$, to obtain the overall stress $\sigma = (\epsilon + \tau_\epsilon\,\dot{\epsilon})/\delta J$. To make the standard linear solid, we add a spring in series. This implies that the stress in the spring will be the same as that in the Kelvin-Voigt solid. Let us label the strain in the Kelvin-Voigt solid as ϵ_{KV} and the strain in the spring as ϵ_{S}. then we have $\sigma = (\epsilon_{\mathrm{KV}} + \tau_\epsilon\,\dot{\epsilon}_{\mathrm{KV}})/\delta J$, $\sigma = \epsilon_{\mathrm{S}}/J_{\mathrm{u}}$, and the total strain $\epsilon = \epsilon_{\mathrm{KV}} + \epsilon_{\mathrm{S}}$. Eliminating ϵ_{S}, we have $\sigma = (\epsilon - \epsilon_{\mathrm{KV}})/J_{\mathrm{u}}$, such that $\epsilon_{\mathrm{KV}} = \epsilon - J_{\mathrm{u}}\,\sigma$. Eliminating ϵ_{KV}, we have $\sigma = (\epsilon - J_{\mathrm{u}}\,\sigma + \tau_\epsilon\,\dot{\epsilon} - \tau_\epsilon\,J_{\mathrm{u}}\,\dot{\sigma})/\delta J$, which may be rearranged in the form

$$\dot{\sigma} + \sigma/\tau_\sigma = M_{\mathrm{u}}\,(\dot{\epsilon} + \tau_\epsilon^{-1}\,\epsilon)$$

where we used $M_{\mathrm{u}} = 1/J_{\mathrm{u}}$, $\delta J = J_{\mathrm{r}} - J_{\mathrm{u}}$, and $J_{\mathrm{r}}/J_{\mathrm{u}} = \tau_\epsilon/\tau_\sigma$.

Problem 5.2.





Frequency variation of the real modulus $M_1(\omega)$ (*top*) and the complex modulus $M_2(\omega)$ (*bottom*) of a standard linear solid on a logarithmic scale in the range $\omega\,\tau_\epsilon = 1/10$–$10$. The plot for Q^{-1} looks similar to M_2 but is centered upon the geometrical mean of the stress and strain relaxation times, $\omega\,\sqrt{\tau_\sigma\tau_\epsilon}$.

Problem 5.3. According to the rules, for a set of $\ell = 1, \ldots, L$ standard linear solids in parallel, all the individual strains are equal, but the individual stresses must be added. Thus, for each SLS, we have

$$\dot{\sigma}_\ell + \sigma_\ell/\tau_\sigma^\ell = M_\mathrm{u}^\ell \left(\dot{\epsilon} + \epsilon/\tau_\epsilon^\ell\right),$$

which takes the frequency-domain form

$$(i\,\omega\,\tau_\sigma^\ell + 1)\sigma_\ell = M_\mathrm{u}^\ell\,\tau_\sigma^\ell\,(i\,\omega\,\tau_\epsilon^\ell + 1)\,\epsilon/\tau_\epsilon^\ell\,.$$

Thus, we have

$$\sigma_\ell = M_\mathrm{u}^\ell\,\tau_\sigma^\ell\,/\tau_\epsilon^\ell\,\frac{1 + i\omega\tau_\epsilon^\ell}{1 + i\omega\tau_\sigma^\ell}\,\epsilon\,.$$

Using $M_\mathrm{r}^\ell/M_\mathrm{u}^\ell = \tau_\sigma^\ell/\tau_\epsilon^\ell$ and summing over ℓ yields (5.29).