Answers to problems

Problem 4.1. Upon making the substitutions $X^I \to x^i$ and $s^I \to s^i$ in (2.388) we find

$$\begin{aligned} E_{ij} &= \tfrac{1}{2}\left[\delta_{ik}\,\partial_j s^k + \delta_{jk}\,\partial_i s^k + (\partial_i s^k)\,\delta_{k\ell}\,(\partial_j s^\ell)\right] \\ &= \tfrac{1}{2}\left[\partial_j s_i + \partial_i s_j + (\partial_i s_\ell)\,(\partial_j s^\ell)\right]. \end{aligned}$$

Expressed in tensor notation, this becomes (4.1).

Problem 4.2. Upon taking the trace of (4.11) we find

$$\begin{aligned} \mathrm{tr}(\mathbf{d}) &= \mathrm{tr}(\boldsymbol{\epsilon}) - \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\epsilon})\,\mathrm{tr}(\mathbf{I}) \\ &= \mathrm{tr}(\boldsymbol{\epsilon}) - \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\epsilon})\,3 \\ &= 0\,. \end{aligned}$$

Problem 4.3. We have

$$\begin{aligned} \Gamma^{ijmn}\,\Gamma^{-1}_{mnk\ell} &= \left[(\kappa - \tfrac{2}{3}\,\mu)\,\delta^{ij}\,\delta^{mn} + \mu\,(\delta^{im}\,\delta^{jn} + \delta^{in}\,\delta^{jm})\right] \\ &\quad \left[\left(\frac{1}{9\,\kappa} - \frac{1}{6\,\mu}\right)\delta_{mn}\,\delta_{k\ell} + \frac{1}{4\,\mu}\,(\delta_{mk}\,\delta_{n\ell} + \delta_{m\ell}\,\delta_{nk})\right] \\ &= (\kappa - \tfrac{2}{3}\,\mu)\left(\frac{1}{9\,\kappa} - \frac{1}{6\,\mu}\right)\delta_{mn}\,\delta_{k\ell}\,\delta^{ij}\,\delta^{mn} \\ &\quad + \mu\left(\frac{1}{9\,\kappa} - \frac{1}{6\,\mu}\right)\delta_{mn}\,\delta_{k\ell}\,(\delta^{im}\,\delta^{jn} + \delta^{in}\,\delta^{jm}) \\ &\quad + (\kappa - \tfrac{2}{3}\,\mu)\,\frac{1}{4\,\mu}\,(\delta_{mk}\,\delta_{n\ell} + \delta_{m\ell}\,\delta_{nk})\,\delta^{ij}\,\delta^{mn} \\ &\quad + \mu\,\frac{1}{4\,\mu}\,(\delta_{mk}\,\delta_{n\ell} + \delta_{m\ell}\,\delta_{nk})\,(\delta^{im}\,\delta^{jn} + \delta^{in}\,\delta^{jm}) \\ &= (\kappa - \tfrac{2}{3}\,\mu)\left(\frac{1}{3\,\kappa} - \frac{1}{2\,\mu}\right)\delta_{k\ell}\,\delta^{ij} + \tfrac{2}{3}\,\mu\left(\frac{1}{3\,\kappa} - \frac{1}{2\,\mu}\right)\delta_{k\ell}\,\delta^{ij} \\ &\quad + (\kappa - \tfrac{2}{3}\,\mu)\,\frac{1}{2\,\mu}\,\delta_{k\ell}\,\delta^{ij} + \frac{1}{2}\,(\delta^i{}_k\,\delta^j{}_\ell + \delta^i{}_\ell\,\delta^j{}_k) \\ &= \left(\frac{1}{3} - \frac{\kappa}{2\,\mu}\right)\delta_{k\ell}\,\delta^{ij} + \left(\frac{\kappa}{2\,\mu} - \frac{1}{3}\right)\delta_{k\ell}\,\delta^{ij} \\ &\quad + \frac{1}{2}\,(\delta^i{}_k\,\delta^j{}_\ell + \delta^i{}_\ell\,\delta^j{}_k) \\ &= \frac{1}{2}\,(\delta^i{}_k\,\delta^j{}_\ell + \delta^i{}_\ell\,\delta^j{}_k)\,. \end{aligned}$$

Problem 4.4. In a fluid, the shear modulus μ vanishes, which implies that the elastic tensor (4.9) takes the form

$$\Gamma^{ijk\ell} = \kappa\,\delta^{ij}\,\delta^{k\ell}\,.$$

Substitution of this elastic tensor in Hooke's law (4.6) yields

$$T^{ij} = \kappa\,\delta^{ij}\,\nabla_k s^k\,.$$

Since in a fluid $T^{ij} = -p\,\delta^{ij}$, this result leads to the acoustic constitutive relationship (4.15).
Problem 4.5. In index notation, the Lagrangian density (4.17) takes the form

$$\tilde{L}(\partial_t s^i, \nabla_i s^j) = \tfrac{1}{2}\,\rho\,\delta_{ij}\,\partial_t s^i \partial_t s^j - \tfrac{1}{2}\,\epsilon_{ij}\,\Gamma^{ijk\ell}\,\epsilon_{k\ell}\,.$$

Thus we have

$$\frac{\partial \tilde{L}}{\partial \partial_t s^i} = \rho\,\delta_{ij}\,\partial_t s^j\,,$$

$$\frac{\partial \tilde{L}}{\partial \nabla_i s^j} = -\,\delta_{jm}\,\Gamma^{imk\ell}\,\epsilon_{k\ell}\,,$$

and the Euler-Lagrange equation (4.18) and boundary condition (4.19) become

$$\rho\,\partial_t^2 s^i = \nabla_j(\Gamma^{ijk\ell}\,\epsilon_{k\ell})\,,$$

$$[\hat{n}_j\,\Gamma^{ijk\ell}\,\epsilon_{k\ell}]_-^+ = 0\,.$$

The tensor form of these equations is as advertised.
Problem 4.6. Upon substitution of the plane wave (4.25) into the isotropic elastic wave equation (4.20), we find

$$\begin{aligned}
-\,\rho\,\omega^2\,\mathbf{a} \;&=\; -\,(\kappa - \tfrac{2}{3}\,\mu)\,\mathbf{k}\,(\mathbf{k}\cdot\mathbf{a}) - \mu\,\mathbf{k}\,(\mathbf{k}\cdot\mathbf{a}) - \mu\mathbf{k}\cdot\mathbf{k}\,\mathbf{a}\\
&=\; -\,(\kappa + \tfrac{1}{3}\,\mu)\,\mathbf{k}\,(\mathbf{k}\cdot\mathbf{a}) - \mu\,\mathbf{a}\,(\mathbf{k}\cdot\mathbf{k})\\
&=\; -\,(\kappa + \tfrac{4}{3}\,\mu)\,\mathbf{k}\,(\mathbf{k}\cdot\mathbf{a}) - \mu\,[\mathbf{a}\,(\mathbf{k}\cdot\mathbf{k}) - \mathbf{k}\,(\mathbf{k}\cdot\mathbf{a})]\,.
\end{aligned}$$

Upon dividing by ρ we find

$$\begin{aligned}
c^2\,\mathbf{a} \;&= \alpha^2\,\hat{\mathbf{k}}\,(\hat{\mathbf{k}}\cdot\mathbf{a}) + \beta^2\,[\mathbf{a}\,(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}) - \hat{\mathbf{k}}\,(\hat{\mathbf{k}}\cdot\mathbf{a})]\\
&= \mathbf{B}\cdot\mathbf{a}\,,
\end{aligned}$$

where $c = \omega/k$, and where $\mathbf{B}$ is given by (4.27).
Problem 4.7. Continuity of traction as expressed by equation (4.22) implies at a fluid solid boundary that

$$-\,p_{\text{fluid}}\,\hat{\mathbf{n}} = \hat{\mathbf{n}}\cdot\mathbf{T}_{\text{solid}}\,,$$

where we have used the fact that in a fluid $\mathbf{T}_{\text{fluid}} = -\,p_{\text{fluid}}\,\mathbf{I}$. Upon dotting this expression with the unit outward normal to the fluid-solid boundary, $\hat{\mathbf{n}}$, we obtain the desired result.
Problem 4.8. Upon substitution of the representation (4.36) into equation (4.32), we find

$$\partial_t^2 \boldsymbol{\nabla}\chi = -\,\boldsymbol{\nabla}p\,,$$

which implies the identity (4.38). Upon taking the divergence of definition (4.36) we find from the acoustic constitutive relationship (4.15) that

$$p = -\,\kappa\,\boldsymbol{\nabla}\cdot(\rho^{-1}\,\boldsymbol{\nabla}\chi)\,,$$

and using equation (4.38) on the left-hand side yields the wave equation (4.37) for the fluid potential χ .

Problem 4.9. Upon substitution of the body force (4.44) into the wave equation (4.42) we find

$$\begin{aligned}\rho\,\partial_t^2\mathbf{s} &= \boldsymbol{\nabla}\cdot\mathbf{T}-\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}\\ &= \boldsymbol{\nabla}\cdot\mathbf{T}_{\text{true}}\,,\end{aligned}$$

where in the last equality we used the definition of the stress glut (4.40). Similarly, substitution of the traction (4.44) into the boundary condition (4.43) yields

$$\mathbf{T}\cdot\hat{\mathbf{n}}=\hat{\mathbf{n}}\cdot\boldsymbol{\gamma}\,,$$

which implies that

$$(\mathbf{T}-\boldsymbol{\gamma})\cdot\hat{\mathbf{n}}=\mathbf{T}_{\text{true}}\cdot\hat{\mathbf{n}}=\mathbf{0}\,.$$

Problem 4.10. Upon substitution of the elastic tensor (4.9) into the definition of the moment density tensor (4.58) we find

$$\begin{aligned}m^{ij} &= \Gamma^{ijk\ell}\,\hat{n}_k\,\Delta s_\ell\\ &= [(\kappa-\tfrac{2}{3}\,\mu)\,\delta^{ij}\,\delta^{k\ell}+\mu\,(\delta^{ik}\,\delta^{j\ell}+\delta^{i\ell}\,\delta^{jk})]\,\hat{n}_k\,\Delta s_\ell\\ &= (\kappa-\tfrac{2}{3}\,\mu)\,\delta^{ij}\,\hat{n}^k\,\Delta s_k+\mu\,(\hat{n}^i\,\Delta s^j+\hat{n}^j\,\Delta s^i)\,.\end{aligned}$$

Since the slip direction is perpendicular to the fault normal, $\hat{n}^k\,\Delta s_k=0$, which leads to expression (4.61).

Problem 4.11. In the weak form, we have the contribution

$$\begin{aligned}\int_V \mathbf{T}_{\text{true}}\!:\!\boldsymbol{\nabla}\tilde{\mathbf{s}}\,dx^3 &= \int_V(\mathbf{T}-\boldsymbol{\gamma})\!:\!\boldsymbol{\nabla}\tilde{\mathbf{s}}\,dx^3\\ &= \int_V \mathbf{T}\!:\!\boldsymbol{\nabla}\tilde{\mathbf{s}}\,dx^3+\int_V\boldsymbol{\gamma}\!:\!\boldsymbol{\nabla}\tilde{\mathbf{s}}\,dx^3\,.\end{aligned}$$

where we used the stress glut (4.40). Upon substitution of the stress glut for an ideal fault (4.57), we conclude that the source contribution is

$$\int_V\boldsymbol{\gamma}\!:\!\boldsymbol{\nabla}\tilde{\mathbf{s}}\,dx^3=\int_\Sigma\mathbf{m}\!:\!\boldsymbol{\nabla}\tilde{\mathbf{s}}\,dx^2\,.$$

Problem 4.12. Upon substituting the body and surface force densities (4.59) into (4.80) we find in components

$$\begin{aligned}s_i(\mathbf{x},t) &= -\int_{-\infty}^{t}\int_V G_{ik}(\mathbf{x},\mathbf{x}';t-t')\,\partial_j'(m_{kj}'\,\delta_\Sigma)\,d\mathrm{V}'\,dt'\\ &\qquad+\int_{-\infty}^{t}\int_{\partial V} G_{ik}(\mathbf{x},\mathbf{x}';t-t')\,\hat{n}_j'\,m_{jk}'\,\delta_\Sigma\,d\mathrm{S}'\,dt'\\ &= -\int_{-\infty}^{t}\int_V G_{ik}(\mathbf{x},\mathbf{x}';t-t')\,\partial_j'(m_{kj}'\,\delta_\Sigma)\,d\mathrm{V}'\,dt'\\ &\qquad+\int_{-\infty}^{t}\int_V \partial_j'[G_{ik}(\mathbf{x},\mathbf{x}';t-t')\,m_{kj}'\,\delta_\Sigma]\,d\mathrm{V}'\,dt'\\ &= \int_{-\infty}^{t}\int_V \partial_j' G_{ik}(\mathbf{x},\mathbf{x}';t-t')\,m_{kj}'\,\delta_\Sigma\,d\mathrm{V}'\,dt'\\ &= \int_{-\infty}^{t}\int_\Sigma \partial_j' G_{ik}(\mathbf{x},\mathbf{x}';t-t')\,m_{kj}(\mathbf{x}',t')\,d\mathrm{S}'\,dt'\,.\end{aligned}$$

Problem 4.13. Upon substitution of equation (4.88) in the weak source (4.63) we find

$$\int_\Sigma \mathbf{m}:\boldsymbol{\nabla}\tilde{\mathbf{s}}\,dx^2 = \mathbf{M}:\boldsymbol{\nabla}\tilde{\mathbf{s}}(\mathbf{x}_\mathrm{s})\,H(t-t_\mathrm{s})\,.$$

Problem 4.14. In index notation, equation (4.173) implies

$$\partial_i\Phi = G\int_V \frac{\rho(x')\,(x_i-x_i')}{\|\mathbf{x}-\mathbf{x}'\|^3}\,d^3x'\,.$$

Thus, we have

$$\partial_i\partial_j\Phi = G\int_V \frac{\rho(x')\,\delta_{ij}}{\|\mathbf{x}-\mathbf{x}'\|^3} - \frac{3\,\rho(x')\,(x_i-x_i')(x_j-x_j')}{\|\mathbf{x}-\mathbf{x}'\|^5}\,d^3x'\,,$$

in agreement with (4.174).
Problem 4.15. In index notation, equation (4.176) is

$$\psi = -\tfrac{1}{2}\left[\Omega_j\,\Omega_j\,x_k\,x_k - (\Omega_j\,x_j)^2\right]\,.$$

Thus we have

$$\begin{aligned}\partial_i\psi \;&=\; -\tfrac{1}{2}\,(2\,\Omega_j\,\Omega_j\,x_k\,\delta_{ik} - \Omega_j\,\delta_{ij}\,\Omega_k\,x_k - \Omega_j\,x_j\,\Omega_k\,\delta_{ik})\\ &=\; -\,\Omega_j\,\Omega_j\,x_i + \Omega_i\,\Omega_j\,x_j\\ &= \epsilon_{ijk}\,\epsilon_{k\ell m}\,\Omega_j\,\Omega_\ell\,x_m\,,\end{aligned}$$

and

$$\partial_i\partial_j\psi \;=\; -\,\Omega_k\,\Omega_k\,\delta_{ij} + \Omega_i\,\Omega_j\,,$$

in agreement with equations (4.177) and (4.178).
Problem 4.16. We have

$$P_2(\cos\theta) = \tfrac{1}{2}\,(3\,\cos^2\theta - 1)\,,$$

and substitution in equation (4.199) yields the desired equality.
Problem 4.17. Substitution of $\Phi = \delta\Phi(r)P_2(\cos\theta)$ and $\rho = \delta\rho(r)\,P_2(\cos\theta)$ into Poisson's equation (4.169) in spherical coordinates, using the Legendre equation at degree 2, namely,

$$\frac{d^2P_2}{d\theta^2} + \cot\theta\,\frac{dP_2}{d\theta} + 6P_2 = 0\,,$$

yields

$$\frac{1}{r^2}\,\frac{d}{dr}\left(r^2\,\frac{d\delta\Phi}{dr}\right) - \frac{6}{r^2}\,\delta\Phi = 4\,\pi\,G\,r\,\delta\rho\,,$$

and substitution of equation (4.201) on the right-hand side yields the desired form. Integration of equation (4.203) over a "pill box" that spans an internal discontinuity at radius $r=d$, usingf continuity of $\delta\Phi$, leads to the second boundary condition in (4.204).
Problem 4.18. In index notation, equation (4.227) becomes

$$\rho^0\,U \;= T^0_{ij}\,E_{ij} + \tfrac{1}{2}\,E_{ij}\,\Xi_{ijk\ell}\,E_{k\ell}\,,$$

and the Lagrangian strain (4.1) is

$$E_{ij} = \tfrac{1}{2}\left[\partial_i s_j + \partial_j s_i + (\partial_i s_k)\,(\partial_j s_k)\right] .$$

Upon substitution, we find to second-order in the gradient of the displacement field that

$$\rho^0\, U \;= T^0_{ij}\,\epsilon_{ij} + \tfrac{1}{2}\,T^0_{ij}\,(\partial_i s_k)\,(\partial_j s_k) + \tfrac{1}{2}\,\epsilon_{ij}\,\Xi_{ijk\ell}\,\epsilon_{k\ell}\,,$$

where $\epsilon_{ij} = \frac{1}{2}(\partial_i s_j + \partial_j s_i)$. The tensor form of this expression is the second expression in equation (4.227).

Problem 4.19. Upon substitution of equation (4.228) into equation (4.229), we find in index notation

$$\begin{aligned}
\rho^0\, U \;&= T^0_{ij}\,\epsilon_{ij} + \tfrac{1}{2}\,(\partial_j s_i)\,\Lambda_{ijk\ell}\,(\partial_\ell s_k)\\
&= T^0_{ij}\,\epsilon_{ij} + \tfrac{1}{2}\,(\partial_j s_i)\,(\Xi_{ijk\ell} + T^0_{\ell j}\,\delta_{ik})\,(\partial_\ell s_k)\\
&= T^0_{ij}\,\epsilon_{ij} + \tfrac{1}{2}\,\epsilon_{ij}\,\Xi_{ijk\ell}\,\epsilon_{k\ell} + \tfrac{1}{2}\,(\partial_j s_i)\,T^0_{\ell j}\,(\partial_\ell s_i)\,,
\end{aligned}$$

in agreement with the second expression in equation (4.227).

Problem 4.20. First, we note that

$$\begin{aligned}
G\int_V \frac{\rho(\mathbf{x})\,\rho(\mathbf{x}')\,\mathbf{s}\cdot(\mathbf{x}-\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|^3}\,dV' \;&= \rho\,\mathbf{s}\cdot\, G\int_V \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|^3}\,dV'\\
&= -\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\Phi
\end{aligned}$$

where we used (4.173) in the last equality. Next, we have

$$\begin{aligned}
\tfrac{1}{2}\,G\int_V \rho(\mathbf{x})\,\rho(\mathbf{x}')\,\mathbf{s}\cdot\boldsymbol{\Pi}(\mathbf{x},\mathbf{x}')\cdot\mathbf{s}'\;dV' \;&= \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot G\int_V \rho(\mathbf{x}')\,\boldsymbol{\Pi}(\mathbf{x},\mathbf{x}')\cdot\mathbf{s}'\;dV'\\
&= -\,\tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\phi
\end{aligned}$$

where we used (4.226) in the last equality. Finally, we have

$$\begin{aligned}
\tfrac{1}{2}\,G\int_V \rho(\mathbf{x})\,\rho(\mathbf{x}')\,\mathbf{s}\cdot\boldsymbol{\Pi}(\mathbf{x},\mathbf{x}')\cdot\mathbf{s}\;dV' \;&= \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot G\int_V \rho(\mathbf{x}')\,\boldsymbol{\Pi}(\mathbf{x},\mathbf{x}')\;dV'\;\cdot\mathbf{s}\\
&= \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla\nabla}\Phi\cdot\mathbf{s}\,,
\end{aligned}$$

where in the last equality we used (4.174). Using these results, we may rewrite (4.242) in the form

$$\begin{aligned}
\tilde{L}(\mathbf{s},\partial_t\mathbf{s},\boldsymbol{\nabla}\mathbf{s}) \;&= \tfrac{1}{2}\,\rho\,\|\partial_t\mathbf{s} + \boldsymbol{\Omega}\times(\mathbf{x}+\mathbf{s})\|^2 - \tfrac{1}{2}\,\rho\,\|\boldsymbol{\Omega}\times\mathbf{x}\|^2\\
&\quad - \boldsymbol{\sigma}^0\!:\!\boldsymbol{\varepsilon} - \tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s})^t\!:\!\boldsymbol{\Lambda}\!:\!(\boldsymbol{\nabla}\mathbf{s})^t\\
&\quad - \rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\Phi - \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\phi - \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla\nabla}\Phi\cdot\mathbf{s}\,.
\end{aligned}$$

We have

$$\begin{aligned}
\|\partial_t\mathbf{s} + \boldsymbol{\Omega}\times(\mathbf{x}+\mathbf{s})\|^2 - \|\boldsymbol{\Omega}\times\mathbf{x}\|^2 \;&= \partial_t\mathbf{s}\cdot\partial_t\mathbf{s} - 2\,(\mathbf{x}+\mathbf{s})\cdot(\boldsymbol{\Omega}\times\partial_t\mathbf{s})\\
&\quad + 2\,(\boldsymbol{\Omega}\times\mathbf{s})\cdot(\boldsymbol{\Omega}\times\mathbf{x}) + \|\boldsymbol{\Omega}\times\mathbf{s}\|^2\\
&= \partial_t\mathbf{s}\cdot\partial_t\mathbf{s} - 2\,(\mathbf{x}+\mathbf{s})\cdot(\boldsymbol{\Omega}\times\partial_t\mathbf{s}) - 2\,\mathbf{s}\cdot\boldsymbol{\nabla}\psi - \mathbf{s}\cdot\boldsymbol{\nabla\nabla}\psi\cdot\mathbf{s}\,,
\end{aligned}$$

where in the last equality we used (4.177) and (4.178). Combining the last two expressions yields the second equality in expression (4.242).

Problem 4.21. We have

$$
\begin{aligned}
\tilde{L}(\mathbf{s}, \partial_t\mathbf{s}, \boldsymbol{\nabla}\mathbf{s}) \; &= \tfrac{1}{2}\,\rho\,[\partial_t\mathbf{s}\cdot\partial_t\mathbf{s} - 2\,(\mathbf{x}+\mathbf{s})\cdot(\boldsymbol{\Omega}\times\partial_t\mathbf{s}) - \mathbf{s}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\psi\cdot\mathbf{s} - 2\,\mathbf{s}\cdot\boldsymbol{\nabla}\psi] \\
&\quad - \boldsymbol{\sigma}^0 : \boldsymbol{\varepsilon} - \tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s})^t : \boldsymbol{\Lambda} : (\boldsymbol{\nabla}\mathbf{s})^t \\
&\quad - \rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\Phi - \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\phi - \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\Phi\cdot\mathbf{s} \\
&= \tfrac{1}{2}\,\rho\,[\partial_t\mathbf{s}\cdot\partial_t\mathbf{s} - 2\,(\mathbf{x}+\mathbf{s})\cdot(\boldsymbol{\Omega}\times\partial_t\mathbf{s})] \\
&\quad - \boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}^0\cdot\mathbf{s}) + \mathbf{s}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^0) - \tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s})^t : \boldsymbol{\Lambda} : (\boldsymbol{\nabla}\mathbf{s})^t \\
&\quad - \rho\,\mathbf{s}\cdot\boldsymbol{\nabla}(\Phi+\psi) - \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\phi - \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}(\Phi+\psi)\cdot\mathbf{s} \\
&= \tfrac{1}{2}\,\rho\,[\partial_t\mathbf{s}\cdot\partial_t\mathbf{s} - 2\,(\mathbf{x}+\mathbf{s})\cdot(\boldsymbol{\Omega}\times\partial_t\mathbf{s})] \\
&\quad - \boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}^0\cdot\mathbf{s}) - \tfrac{1}{2}\,(\boldsymbol{\nabla}\mathbf{s})^t : \boldsymbol{\Lambda} : (\boldsymbol{\nabla}\mathbf{s})^t \\
&\quad - \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\phi - \tfrac{1}{2}\,\rho\,\mathbf{s}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}(\Phi+\psi)\cdot\mathbf{s}\,,
\end{aligned}
$$

where we have used the equilibrium condition (4.182). The term $\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}^0\cdot\mathbf{s})$ integrates out using vanishing of the surface traction, $\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}^0 = \mathbf{0}$.

Problem 4.22. Variation of the term $-2\,\rho\,\mathbf{x}\cdot(\boldsymbol{\Omega}\times\partial_t\mathbf{s})$ with respect to $\partial_t\mathbf{s}$ yields the time-independent term $2\,\rho\,\boldsymbol{\Omega}\times\mathbf{x}$. Differentiation of this term with respect to time in the Euler-Lagrange equations makes it vanish.

Problem 4.23. We have by exchanging i and k in (4.271)

$$
\begin{aligned}
\rho\,B_{ki} \; &= \Lambda_{kji\ell}\,\hat{k}_j\,\hat{k}_\ell \\
&= \Lambda_{k\ell ij}\,\hat{k}_\ell\,\hat{k}_j \\
&= \Lambda_{ijk\ell}\,\hat{k}_j\,\hat{k}_\ell \\
&= \rho\,B_{ik}\,.
\end{aligned}
$$

where in the third equality we used the symmetry (4.230).

Problem 4.24. Substitution of equation (4.276) yields

$$
\begin{aligned}
&\nabla_j(\Lambda_{ijk\ell}\nabla_\ell s_k) - \rho\,s_j\nabla_j\nabla_i(\Phi+\psi) \\
&= \; \nabla_j(\Gamma_{ijk\ell}\,\epsilon_{k\ell}) - (\nabla_i p^0)\,\nabla_k s_k + (\nabla_j p^0)\,\nabla_i s_j \\
&\quad\; - \nabla_j[\rho\,s_j\,\nabla_i(\Phi+\psi)] + \nabla_j(\rho\,s_j)\,\nabla_i(\Phi+\psi) \\
&\quad\; + \nabla_j[\tfrac{1}{2}\,(\tau^0_{ij}\,\epsilon_{kk} + \tau^0_{k\ell}\,\delta_{ij}\epsilon_{k\ell}) - \tau^0_{ik}\,\epsilon_{jk} + \tau^0_{jk}\,\omega_{ik}] \\
&= \; \nabla_j(\Gamma_{ijk\ell}\,\epsilon_{k\ell}) - \nabla_j(s_j\,\nabla_i p^0) + \nabla_i(s_j\nabla_j p^0) \\
&\quad\; - \nabla_j[\rho\,s_j\,\nabla_i(\Phi+\psi)] + \nabla_j(\rho\,s_j)\,\nabla_i(\Phi+\psi) \\
&\quad\; + \nabla_j[\tfrac{1}{2}\,(\tau^0_{ij}\,\epsilon_{kk} + \tau^0_{k\ell}\,\delta_{ij}\epsilon_{k\ell}) - \tau^0_{ik}\,\epsilon_{jk} - \tau^0_{jk}\,\omega_{ki}]\,.
\end{aligned}
$$

Now we use the equilibrium condition (4.182), namely,

$$
\nabla_j p^0 = \nabla_k\tau^0_{jk} - \rho\,\nabla_j(\Phi+\psi)\,,
$$

such that

$$
\begin{aligned}
&\nabla_j(\Lambda_{ijk\ell}\nabla_\ell s_k) - \rho\, s_j \nabla_j \nabla_i(\Phi+\psi) \\
&= \nabla_j(\Gamma_{ijk\ell}\,\epsilon_{k\ell}) - \nabla_i[\rho\, s_j\, \nabla_j(\Phi+\psi)] + \nabla_j(\rho\, s_j)\,\nabla_i(\Phi+\psi) + \nabla_i(s_j\,\nabla_k \tau^0_{jk}) \\
&\quad - \nabla_j(s_j\,\nabla_k\tau^0_{ik}) + \nabla_j[\tfrac{1}{2}\,(\tau^0_{k\ell}\,\delta_{ij}\,\epsilon_{k\ell} - \tau^0_{ij}\,\epsilon_{kk}) + \omega_{ik}\,\tau^0_{kj} - \tau^0_{ik}\,\omega_{kj} + \tau^0_{ij}\,\epsilon_{kk} - \tau^0_{ik}\,\nabla_k s_j] \\
&= \nabla_j(\Gamma_{ijk\ell}\,\epsilon_{k\ell}) - \nabla_i[\rho\, s_j\, \nabla_j(\Phi+\psi)] + \nabla_j(\rho\, s_j)\,\nabla_i(\Phi+\psi) + \nabla_i(s_j\,\nabla_k \tau^0_{jk}) \\
&\quad + \nabla_j[\tfrac{1}{2}\,(\tau^0_{k\ell}\,\delta_{ij}\,\epsilon_{k\ell} - \tau^0_{ij}\,\epsilon_{kk}) + \omega_{ik}\,\tau^0_{kj} - \tau^0_{ik}\,\omega_{kj}] \\
&\quad + \nabla_j(\tau^0_{ij}\,\nabla_k s_k - \tau^0_{ik}\,\nabla_k s_j) - \nabla_j(s_j\,\nabla_k\tau^0_{ik}) \\
&= \nabla_j(\Gamma_{ijk\ell}\,\epsilon_{k\ell}) - \nabla_i[\rho\, s_j\, \nabla_j(\Phi+\psi)] + \nabla_j(\rho\, s_j)\,\nabla_i(\Phi+\psi) \\
&\quad + \nabla_j[\tfrac{1}{2}\,(\tau^0_{k\ell}\,\delta_{ij}\,\epsilon_{k\ell} - \tau^0_{ij}\,\epsilon_{kk}) + \omega_{ik}\,\tau^0_{kj} - \tau^0_{ik}\,\omega_{kj}] \\
&\quad + \nabla_i(s_j\,\nabla_k\tau^0_{jk}) - \nabla_j(s_k\,\nabla_k\tau^0_{ij})\,,
\end{aligned}
$$

as required.